On History and Humility

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Mathematics of the 19th Century: Geometry/ Analytic Function Theory, Edited by A.N. Kolmogorov and A. P. Yushkevich, Translated from the Russian by Roger Cooke, Birkhäuser Verlag, 1996, Basel. pp. x+290.

For a subject that depends so heavily on its past, mathematics pays surprisingly little attention to its history. Almost every practicing mathematician frequently uses mathematics from the 19th century. Most do so unwittingly, and few know the context in which that mathematics was crafted. We are poorer for it.

This is the second volume in a series on the history of *Mathematics of the 19th Century*, originally published in Russian and now translated into English. It's not broad, sweeping history that sets the mathematics in context. There is little mention of connections to modern mathematics at all. Rather the authors thread together snippets, and the reader glimpses a century of mathematics as a series of flashes – facts and stories and curiosities. In that sense, readers must work to access the benefits of history.

This volume includes Geometry and Analytic Functions. It's an ambitious task – more ambitious than the first volume, which covered logic, algebra, number theory, and probability. Both Geometry and Function Theory began the century as small subjects, rather naïve in outlook and goals. Both ended the century as part of a glorious century of dramatic mathematical growth; they reached across great expanses of mathematics and influenced not just the rest of mathematics but much of science as well. We can see in the development of these two areas over a century the shadow of the development of all mathematics over a millennium.

What does one learn by reading about history? One learns culture. One learns perspective. One learns humility.

The culture is easy to explain. Stories of mathematicians and mathematics are tucked between passages of dry mathematical explication. Here's a small sample.

How did the term "topology" arise? On page 99, the authors write:

> The term "topology" first appeared in "Vorstundien zur Topologie (*Göttingen Studndien, 1847*), written by Johann Benedikt Listing (1808-1882), a mathematician and physicist at Göttingen and a student of Gauss. This term, now universally accepted but rarely applied until the 1920's, was a neologism coined by Listing himself; Listing proposed using this term, compounded from the Greek words τοπος(place) and λογος(study), as a replacement for Leibniz' term

> *geometria situs* , since the word *geometry* suggested measurement, which plays no role in topology.

Where did the Möbius band first appear? The authors tell us, and add some color about its reception.

> Möbius first described the "Möbius band" in a paper presented to the Paris Academy in 1861 as an entry to a competition on the theme "Improve in some important point the geometric theory of polyhedra." Möbius' paper, written in bad French and containing many new ideas, was not understood by the joury, and like the other papers submitted to the competition, was not awarded the prize.

Who was first to recognize multi-valued complex functions? It was Gauss of course, and much earlier than most imagine. This is more serious culture than the previous examples, and well worth knowing. In December of 1811, Gauss wrote a long letter to Bessel.

> . . . What is meant by $\int \varphi(x)\, dx$ for $x = a + bi$? Obviously if we wish to start from clear concepts, we must assume that x starts from a value for which the integral must equal zero and passes to $x = a + bi$ through infinitesimal increments . . . Thus the meaning is completely established. But the passage may take place in infinitely many ways. . . . I now assert that the integral $\int \varphi(x)\, dx$ over two different paths preserves the same value if inside the portion of the plane enclosed between two lines the function $\varphi(x)$ is never equal to ∞. This is a beautiful theorem whose simple proof I shall give on a suitable occasion. It is connected with other beautiful truths involving series expansions. The passage must always be carried out at each poiint in such a way as never to involve points where $\varphi(x) = \infty$. I insist that such points must be avoided because for them the original fundamental concept of the integral $\int \varphi(x)\, dx$ obviously loses its clarity and easily leads to contradictions.
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> At the same time, however, it is clear from this how the function generated by the integral $\int \varphi(x)\, dx$ can have many values for the same value of x, specifically depending on whether a single or multiple circuit about the point at which $\varphi(x) = \infty$ is allowed or no such circuit is allowed. If, for example, we define log x by $\int dx/x$ starting from $x = 1$, we can arrive at log x either by not going around the point $x = 0$ or by encircling it once or several times; each time the constant $2\pi i$ or $-2\pi i$ will be added. If, however, $\varphi(x)$ does not become infinite for any finite value of x, then the integral is always a single-valued function of x.

History – even the snippet variety of history – teaches us more than culture, however. One of the greatest lessons of history is perspective, and for mathematics perspective is crucial to understanding our own place in the river of mathematics.

Rivers have both length and breadth. Surveying 100 years of a subject such as analytic function theory, starting with only primitive inklings of the notion of complex variables and ending with a its triumph in proving the Prime Number Theorem, helps us to understand the length of that river and how far we have come down stream in the succeeded 100 years. It helps us even more to realize that we yet have far to go.

Surveying 19th century geometry gives us insight into the width of our own river. Fashions came and went during that time – projective geometry, intricate calculations of curvature and torsion, special algebraic curves, pencils and series, Quaternions, and non-euclidean geometries of all kinds. Almost every one of these topics remains an important part of mathematics (I suspect that the authors have screened out those parts that are not), but every one of these topics was a fad of its day, attracting those young mathematicians who wanted to make their mathematical mark for the future. Indeed, we still remember Christian von Staudt, but only for his identity involving Bernoulli numbers and *not* for his acclaimed work on synthetic projective geometry. The acrimonious debates of the mid-nineteenth century about the merits of synthetic versus analytic geometry are largely forgotten today. This ought to help us understand our own place in our own river.

But history's most valuable lesson is humility. I don't mean humility about one's own work (although that often comes from understanding the work of others, whether ancestors or contemporaries.) I mean rather the humility brought eventually to all who predict the future. History's most valuable lesson is that understanding the future is beyond our abilities: Even the *best* mathematicians are unable to predict what mathematics will be like 50 or 100 years from now. There is no reason to believe a law verified by hundreds of years of experience is less true today than in the past.

Humility is particularly important in an age when we try to set forth strategic plans for everything, including research, and when we are challenged to work only on projects with achievable goals.

How do we predict which mathematics will have important applications? There were a few instances when indeed mathematicians set out with applications in mind and achieved far more than their original goals. Early in the 19th century, the local government desired a geodesic map of the Duchy of Hannover. They turned to Gauss, and during the 1820s he set out to solve their problem and created the basis for much of differential geometry for the next 100 years. His mathematics was blended wonderfully with practice, and Gauss set down his theory (in Latin), carried out vast computations, and traipsed about the woods of Lower Saxony – all at the same time.

Far more often, however, the process has worked in the other direction – it was the mathematics that found the applications, and often many years in the future. The work of Gauss on geometry serves as one example. That work was taken up by Jacobi and Minding during the next 50 years, not with applications in mind but because the mathematics was elegant. They provided the germs of geometric ideas long into the future, such as parallel transport (1917) and deformations of surfaces (1950s). Riemann took up the ideas of intrinsic geometry and described higher dimensional manifolds long before anyone suspected they would replace Newton's ether in physics. His work revolutionized geometry in the 19th century, and profoundly affected mathematics and physics in the 20th.

Yet even years after Riemann had died, mathematicians were defending his "useless" abstractions. Poincaré wrote in his famous 1895 memoir *Analysis Situs*:

> "The geometry of *n* dimensions studies reality; no one doubts that. Bodies in hyperspace are subject to precise definitions, just like bodies in ordinary space; and while we cannot draw pictures of them, we can imagine and study them."

And by the mid-twentieth century, nearly a hundred years after Riemann has done his work, Einstein gave the final apology:

> "Only the genius of Riemann, solitary and uncomprehended, had already won its way by the middle of the last century to a new conception of space … in which it power to take part in physical events was recognized as possible."

Reading the history of 19th century geometry and function theory teaches many similar lessons. Who could have anticipated String Theory and computer graphics from the geodesy of Gauss? Who could have foreseen 20th century partial differential equations and computational mathematics in the work of Cauchy, Dirichlet, and Weierstrass? Who could have known that the fussy calculations of Jacobi, Eisenstein, and Jacobi on Elliptic curves and analytic functions would lead first to a proof of the Prime Number Theorem at the end of the 19th century, and then to a proof of the Fermat Conjecture at the end of the 20th? No one knew, of course, and no one pretended to understand the future.

History teaches us humility, but because many mathematicians ignore their history, it's a lesson often ignored. Mathematicians deride "research for research's sake"; departments make fine distinctions between useful and useless mathematics; agencies seek to justify funding our most recent work by citing "breakthroughs" paper by paper (and grant by grant). It is a measure of our arrogance that we believe we are so different from the past – that we see ourselves as the pinnacle of mathematical achievement rather than participants in our history.

Culture, perspective, and humility. Mathematics can profit from more of these in the coming years. Perhaps we should spend more time studying our history and less time creating strategic plans.

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